# Reduced Basis Approximations of Structural Displacements for Optimal Design

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**A new solution procedure for improved approximate reanalyses of structures, using results of a single precise analysis, is presented. The proposed procedure is based on combining the computed terms of a series expansion, used as high-quality basis vectors, with coefficients of a reduced basis expression. The latter coefficients can readily be determined for each trial design by various criteria. The results are measured by the errors in satisfying the analysis equations and are compared in terms of the quality of the approximations. The proposed approach is suitable for various types of design variables and can be used with a general finite element model. A general reanalysis procedure is introduced and the physical significance of first-order approximations is demonstrated. Several numerical examples illustrate the effectiveness of the solution process. It is shown that high-quality approximations of displacements can be obtained with a small computational effort for very large changes in the design variables.**

## Introduction

In most structural optimization problems the implicit behavior constraints must be evaluated for successive modifications in the design. For each trial design the analysis equations must be solved and the multiple repeated analyses usually involve extensive computational effort. Consequently, optimization of large-scale structures might become prohibitive. This difficulty motivated several studies on explicit approximations of the structural behavior (i.e., displacements and stresses) in terms of the design variables.[1-4] Although this approach can considerably reduce the amount of computations, the quality of the approximations might be insufficient. Many of the approximate behavior models proposed in the past are valid only for relatively small changes in the design variables. For large changes in the design, the accuracy of the approximations is deteriorated and might become meaningless. It has been shown that approximations of forces are of higher quality than those of displacements.[5,6]

Several means have been proposed to improve the quality of the approximations. One of the early studies on structural optimization showed that assuming the inverse (reciprocal) cross-sectional areas as design variables might considerably improve the results.[7] Since then, further studies confirmed this property and clarified some of the reasons for this phenomenon.[1,8,9] The inverse variables formulation can be viewed as a special case of the more general approach of using intervening variables.[8,10,11] Another approach is to scale the initial design such that the changes in the design variables are reduced.[5,12,13] It has been shown that the scaling operation is useful for various types of design variables and behavior functions. Recently, this approach has been found most effective for homogeneous functions.[10]

The object of this study is to present a new solution procedure, based on results of a single precise analysis, for improved approximate reanalysis of structures. It will be shown that the quality of displacement approximations can greatly be improved by combining the computed terms of a series expansion, used as high-quality basis vectors, with coefficients of a reduced basis expression. The latter coefficients can readily be determined such that a reduced set of the analysis equations is satisfied. An alternative criterion, based on minimizing the errors in satisfying the analysis equations, is demonstrated. The two criteria are compared in terms of the quality of the approximations. The results are measured by the errors in satisfying the analysis equations, which can readily be determined without performing a precise analysis.

It is further shown that first-order approximations of the proposed model can be introduced by combining scaling of the initial design and scaling of a fictitious set of loads. Integrating these two types of scaling, the approximate displacements can be expressed in a reduced basis form as functions of two coefficients. A general reanalysis procedure for practical structures is introduced and the physical significance of first-order approximations is demonstrated.

The proposed approach is suitable for various types of design variables and can be used with a general finite element model. Several numerical examples illustrate the solution methodology and the effectiveness of the solution process. It is shown that high-quality approximations of displacements can be obtained with a small computational effort for very large changes in the design variables.

## General Formulation

### Problem Statement

The problem under consideration can be stated as follows: Given an initial design variables vector $X^*$, the corresponding stiffness matrix $K^*$, and the displacements $r^*$, computed by the equilibrium equations

$$K^* r^* = R \tag{1}$$

where $R$ is the load vector, whose elements are often assumed to be independent of the design variables. The stiffness matrix $K^*$ is usually given from the initial analysis in the decomposed form

$$K^* = U^{*T} U^* \tag{2}$$

where $U^*$ is an upper triangular matrix. Assume a change $\Delta X$ in the design variables so that the modified design is

$$X = X^* + \Delta X \tag{3}$$

and the corresponding stiffness matrix is

$$K = K^* + \Delta K \tag{4}$$

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where $\Delta K$ is the change in the stiffness matrix due to the change $\Delta X$.

The object is to find efficient and high-quality approximations of the modified displacements $r$ due to various changes in the design variables $\Delta X$, without solving the modified analysis equations

$$Kr = (K^* + \Delta K)r = R \tag{5}$$

The elements of the stiffness matrix are not restricted to certain forms and can be general functions of the design variables. That is, the design variables $X$ may represent coordinates of joints, the structural shape, geometry, members cross sections, etc.

The stresses $\sigma$ are given explicitly in terms of the computed displacements $r$ by

$$\sigma = Sr \tag{6}$$

where $S$ is the system stress transformation matrix. Once the displacements are calculated, the stresses can readily be determined by Eq. (6). Thus, the proposed approximations of $r$ are intended only to replace the set of implicit equations [Eq. (5)].

### Series Approximations

A common approach is to consider the first terms of a series expansion, to obtain the approximate displacements $r_a$

$$r_a = r_1 + r_2 + r_3 \cdots \tag{7}$$

Taylor series expansion is one of the most commonly used approximations in structural optimization. The first three terms, obtained by expanding $r$ about $X^*$, are given by

$$\begin{aligned} r_1 &= r^* \\ r_2 &= r_x^* \Delta X \\ r_{3j} &= 1/2\ \Delta X^T H_j^* \Delta X \end{aligned} \tag{8}$$

where the displacements $r^*$, the matrix of first derivatives $r_x^*$, and the matrix of second derivatives of $r_j$, $H_j^*$, are computed at $X^*$. $r_{3j}$ is the jth component of $r_3$.

An alternative series is obtained by rearranging Eq. (5) to read

$$K^* r = R - \Delta K r \tag{9}$$

Writing this equation as the recurrence relation

$$K^* r^{(k+1)} = R - \Delta K r^{(k)} \tag{10}$$

and assuming the intial value $r^{(1)} = r^*$, the following Binomial series expansion is obtained

$$r_a = (I - B + B^2 - \cdots)\, r^* \tag{11}$$

where $B$ is defined by

$$B = K^{*-1} \Delta K \tag{12}$$

That is, the first three terms of the series are given by

$$\begin{aligned} r_1 &= r^* \\ r_2 &= -Br^* \\ r_3 &= B^2 r^* \end{aligned} \tag{13}$$

Calculation of $r_a$ by Eq. (11) involves only forward and backward substitutions if $K^*$ is given in the decomposed form of Eq. (2). It can be shown that the terms of Eq. (13) are equivalent to those of Taylor series [Eq. (8)] for some homogeneous displacement functions.

## Improved Approximations

### Reduced Basis Approach

The reduced basis approach is based on evaluation of the displacements in terms of a reduced number ($n$) of basis vectors $r_1, r_2, \ldots, r_n$ by

$$r_a = y_1 r_1 + y_2 r_2 + \cdots + y_n r_n = r_b y \tag{14}$$

where

$$r_b = \{r_1, r_2, \ldots, r_n\} \tag{15}$$

and $y$ is a vector of coefficients to be determined

$$y^T = \{y_1, y_2, \ldots, y_n\} \tag{16}$$

Substituting Eq. (14) into the analysis equations [Eq. (5)] and premultiplying by $r_b^T$ yields[14]

$$r_b^T K r_b y = r_b^T R \tag{17}$$

That is, $y$ is determined by solving the reduced set of $n \times n$ equations [Eq. (17)]. The latter equations can be written in the general form

$$ay = b \tag{18}$$

where $a$ and $b$ are given by

$$\begin{aligned} a &= r_b^T K r_b \\ b &= r_b^T R \end{aligned} \tag{19}$$

An alternative reduced basis procedure is to define a fictitious load vector $R_a$

$$R_a = (K^* + \Delta K) r_a = K r_a \tag{20}$$

Note that $r_a$ are precise displacements for the stiffness matrix $K = K^* + \Delta K$ and the fictitious load vector $R_a$. The discrepancy in satisfying the equilibrium equations [Eq. (5)] due to the approximate displacements $r_a$ is given by [see Eqs. (14) and (20)]

$$\Delta R(y) = R_a - R = K r_a - R = K r_b y - R \tag{21}$$

Evidently, if $r_a$ is the vector of precise displacements, then $\Delta R = 0$. Thus, $\Delta R$ can be used to evaluate the quality of the approximations. Define the common measure of smallness of $\Delta R(y)$ by the quadratic form

$$Q(y) = \Delta R^T W \Delta R \tag{22}$$

where $W$ is a predetermined diagonal matrix, giving the relative weight of the components of $\Delta R(y)$. In the absence of particular requirements, a natural choice is $W = I$. Alternatively, the elements of $W$ can be chosen such that the errors in certain equilibrium equations will be minimized. In particular, if only several displacements are of interest, the corresponding elements of $W$ can be selected accordingly.

Substituting Eq. (21) into Eq. (22) yields

$$Q(y) = (K r_b y - R)^T W (K r_b y - R) \tag{23}$$

Differentiating Eq. (23) with respect to $y$ and setting the results equal to zero, we obtain a set of linear equations in the form of Eq. (18), where $a$ and $b$ are given by

$$\begin{aligned} a &= (K r_b)^T W (K r_b) \\ b &= (K r_b)^T W\, R \end{aligned} \tag{24}$$

The reduced basis method is most efficient in cases where the number of basis vectors is much smaller than the number of equilibrium equations. A major problem in using the method is that it is not always clear how to select effectively high-quality basis vectors. Results of several precise analyses may be used for this purpose. The disadvantage of this procedure is that such calculations usually require extensive computational effort. An alternative procedure that is based on results of a single precise analysis is described subsequently.

**General Solution Procedure**

The proposed solution procedure is based on combining the computed terms of the series of [Eq. (7)], used as high-quality basis vectors $\boldsymbol{r}_i$, with selected coefficients $\boldsymbol{y}$ of the reduced basis expression [Eq. (14)], such that the quality of the approximate displacements $\boldsymbol{r}_a$ is improved. That is, the series terms are selected as basis vectors. The procedure is general and different types of design variables may be considered (i.e., geometrical variables, cross-sectional variables, etc.). It can be used with various finite element programs considering different versions, such as:

- Various selections of the basis vectors $\boldsymbol{r}_i$ [i.e., Eq. (8) or Eq. (13)].
- Various criteria for selecting the parameters $\boldsymbol{y}$ [i.e., Eq. (19) or Eq. (24)].

The procedure involves the following steps:

1) The modified stiffness matrix $\boldsymbol{K}$ is introduced.

2) The vectors $\boldsymbol{r}_i$ [Eq. (8) or Eq. (13)] are calculated. To preserve efficiency of the calculations, only two or three basis vectors will be considered.

3) The elements of $\boldsymbol{a}$ and $\boldsymbol{b}$ [Eq. (19) or Eq. (24)] are determined. A comparison between these two criteria will be demonstrated later by several numerical examples.

4) The coefficients $\boldsymbol{y}$ are calculated by solving the set of ($2 \times 2$ or $3 \times 3$) equations [Eq. (18)].

5) The approximate displacements $\boldsymbol{r}_a$ [Eq. (14)] are evaluated.

6) The value of the quadratic function $Q(y)$ [Eq. (23)] is calculated and the quality of the approximations is evaluated. A possible criterion for acceptable approximations is

$$Q(y) \le Q^U \tag{25}$$

where $Q^U$ is a predetermined bound on the errors in satisfying the equilibrium equations. This criterion does not require precise analysis; however, it is possible that better results will be obtained by the criterion of Eq. (19), which provides higher $Q$ value than that of Eq. (24).

Applying this procedure, the computed terms of a series expansion and the coefficients of a reduced basis expression are combined to obtain a powerful solution procedure for efficient and high-quality approximations. The effectiveness of the solution process will be demonstrated later by several numerical examples.

In summary, the proposed procedure can be viewed as:

1) a reduced basis approach where series expansion terms are used as basis vectors; or

2) a modified series expansion where the series terms are modified to include scaling coefficients $\boldsymbol{y}$. The latter are selected such that the quality of the approximations is improved. In the special case where $y = 1.0$ is assumed, the proposed procedure is reduced to conventional series expansion.

The physical significance is illustrated in Fig. 1 where the solution process is shown for a three-bar truss having two unknown displacements. The basis vectors $\boldsymbol{r}_1, \boldsymbol{r}_2, \boldsymbol{r}_3$ are shown in the space of displacements. It can be seen that the conventional Taylor series or the Binomial series (which are equivalent in the present example) $\boldsymbol{r}_a = \boldsymbol{r}_1 + \boldsymbol{r}_2 + \boldsymbol{r}_3$ [Eq. (7)] diverge. Assuming the proposed procedure, the precise solution $\boldsymbol{r} = y_1\boldsymbol{r}_1 + y_2\boldsymbol{r}_2$ is achieved with only two basis vectors.

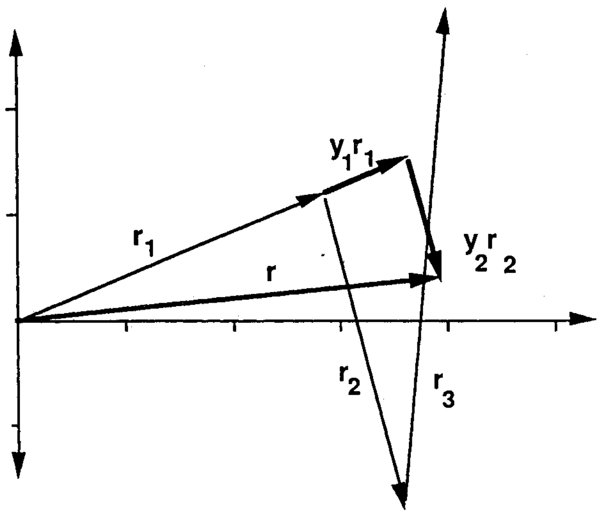


**Fig. 1 Effect of *y* on approximate displacements.**

## First-Order Approximations

**Improved First-Order Approximations**

The solution procedure proposed in this study is general and suitable for any number of basis vectors. However, due to efficiency considerations, first-order approximations are often used. If only two terms of the series [Eq. (7)] are considered, the following first-order approximations of Taylor series [Eq. (8)] and the Binomial series [Eq. (13)] are obtained

$$\boldsymbol{r}_a = \boldsymbol{r}^* + \boldsymbol{r}_x^* \Delta \boldsymbol{X} \tag{26}$$

$$\boldsymbol{r}_a = (\boldsymbol{I} - \boldsymbol{B})\boldsymbol{r}^* \tag{27}$$

The latter equation can also be introduced by substituting

$$\boldsymbol{r} = \boldsymbol{r}^* + \Delta \boldsymbol{r} \tag{28}$$

into the right-hand side of Eq. (9), giving

$$\boldsymbol{K}^*\boldsymbol{r} = \boldsymbol{R} - \Delta\boldsymbol{K}\boldsymbol{r}^* - \Delta\boldsymbol{K}\Delta\boldsymbol{r} \tag{29}$$

Neglecting the second-order term $\Delta\boldsymbol{K}\Delta\boldsymbol{r}$, premultiplying by $\boldsymbol{K}^{*-1}$, and substituting Eqs. (1) and (12) gives Eq. (27).

Considering only the two basis vectors of Eq. (26) or Eq. (27), then Eq. (14) becomes

$$\boldsymbol{r}_a = y_1\boldsymbol{r}_1 + y_2\boldsymbol{r}_2 = \boldsymbol{r}_b\boldsymbol{y} \tag{30}$$

where

$$\boldsymbol{r}_b = \{\boldsymbol{r}_1, \boldsymbol{r}_2\} \tag{31}$$

$$\boldsymbol{y}^T = \{y_1, y_2\} \tag{32}$$

The elements of $\boldsymbol{a}$ and $\boldsymbol{b}$ [Eqs. (19) and (24), respectively] are given by

$$a_{ij} = \boldsymbol{r}_i^T(\boldsymbol{K}\boldsymbol{r}_j) \qquad i = 1,2 \quad j = 1,2$$

$$b_i = \boldsymbol{r}_i^T\boldsymbol{R} \qquad i = 1,2 \tag{33}$$

$$a_{ij} = (\boldsymbol{K}\boldsymbol{r}_i)^T\boldsymbol{W}(\boldsymbol{K}\boldsymbol{r}_j) \qquad i = 1,2 \quad j = 1,2$$

$$b_i = (\boldsymbol{K}\boldsymbol{r}_i)^T\boldsymbol{W}\boldsymbol{R} \qquad i = 1,2 \tag{34}$$

It will be shown now that by combining two types of scaling (scaling of the initial stiffness matrix $\boldsymbol{K}^*$, and scaling of the fictitious set of loads $\boldsymbol{R}_a$), the first-order Binomial series approximations of Eq. (27) can be expressed in the reduced basis form of Eq. (30). A similar procedure can be used for Taylor series approximations.

Scaling of the initial stiffness matrix $K^*$ is defined by

$$K = \mu K^* \tag{35}$$

where $\mu$ is a positive scalar multiplier. From Eqs. (1), (5), and (35) it is clear that the precise displacements after scaling can be calculated directly by

$$r = \mu^{-1} r^* \tag{36}$$

It should be noted that Eq. (35) does not require linear dependence of $K$ on $X$. Furthermore, in many cases of general changes in $K$, the elements of $\mu K^*$ do not correspond to an actual design. That is, the matrix $K$ computed by Eq. (35) does not have the usual physical meaning. Scaling of the initial stiffness matrix [Eq. (35)] will improve the quality of the approximations, if the known modified displacements $\mu^{-1} r^*$ [Eq. (36)] provide better initial data than the original displacements $r^*$.

The modified stiffness matrix $K$ [Eq. (4)] can be expressed in terms of $\mu$ by (see Fig. 2)

$$K = K^* + \Delta K = \mu K^* + \Delta K \mu \tag{37}$$

That is, if an initial design $\mu K^*$ is assumed instead of $K^*$, the modified stiffness matrix $K$ is expressed in terms of the corresponding changes in the stiffness matrix $\Delta K \mu$ instead of $\Delta K$.

Consider now the first-order Binomial approximations of Eq. (27). Based on Eqs. (35) and (36), it is possible to assume $\mu K^*$ and $\mu^{-1} r^*$ as initial values. From Eq. (37), the corresponding changes in the stiffness matrix are

$$\Delta K \mu = (1 - \mu) K^* + \Delta K \tag{38}$$

Substituting $\mu K^*$, $\mu^{-1} r^*$, and $\Delta K_\mu$ [Eq. (38)] into Eq. (27) instead of $K^*$, $r^*$, and $\Delta K$, respectively, yields

$$r_a = \mu^{-2}(2\mu - 1) r^* - \mu^{-2} B r^* \tag{39}$$

It can be noted that for $\mu = 1$, Eq. (27) is obtained.

Several criteria for selecting the value of $\mu$ have been proposed elsewhere.[5,12,13] The scaling of loads presented here is used to introduce a more effective procedure. The fictitious loads $R_a$ can be scaled by

$$R_s = \Omega R_a \tag{40}$$

where $\Omega$ is a scalar. The precise displacements $r_s$ corresponding to the modified stiffness matrix $K$ and the scaled fictitious loads $R_s$ are given by

$$r_s = \Omega r_a \tag{41}$$

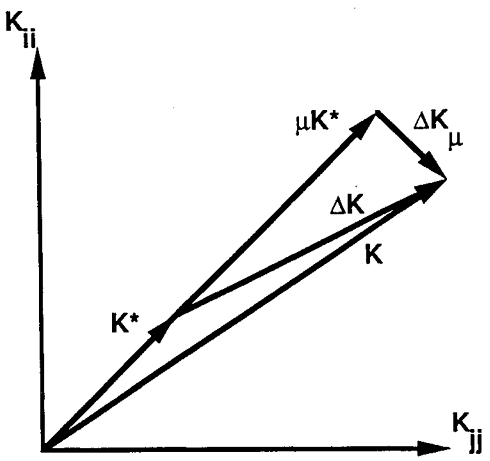


**Fig. 2 Scaling of *K*.**

Evaluating $r_a$ for any given $\mu$ by Eq. (39), the resulting $R_a$ can readily be calculated by Eq. (20). The latter fictitious loads can then be scaled by Eq. (40) such that the final displacements $\Omega\, r_a$ [Eq. (41)] are improved. Substituting Eq. (39) into Eq. (41) yields

$$r_s = \Omega[\mu^{-2}(2\mu - 1) r_1 + \mu^{-2} r_2] \tag{42}$$

The physical significance of the solution process is as follows. Each evaluation of the displacements can be viewed as the following two steps:

1) Selecting $\mu$ by scaling of the initial stiffness matrix $K^*$ and evaluation of the approximate displacements $r_a$ for the given loads $R$.

2) Selecting $\Omega$ by scaling of the fictitious loads $R_a$ and the corresponding displacements $r_a$ for the given modified design $K$.

Assuming the transformation

$$y_1 = \Omega[\mu^{-2}(2\mu - 1)]$$
$$y_2 = \Omega \mu^{-2} \tag{43}$$

and substituting Eqs. (43) into Eq. (42) gives Eq. (30). That is, Eq. (42), which is based on combining the two types of scaling, is equivalent to the two-terms expression of the reduced basis method [Eq. (30)]. It is instructive to note that $\mu$ and $\Omega$ are determined uniquely for any assumed $y$ by Eq. (43).

#### Computational Considerations

In general, the quality of the results and the efficiency of the calculations are two conflicting factors that should be considered in selecting an approximate reanalysis model. That is, better approximations are often achieved at the expense of more computational effort. In this section, some computational considerations associated with the presented approximate models are discussed. Consider first the two methods of calculating the basis vectors.

Assuming the common first-order Taylor series expansion, once the matrix $r_x^*$ is available each redesign involves only calculation of the product $r_x^* \Delta X$. This is probably the most efficient reanalysis model. However, it has been noted that the quality of the results might be insufficient for large changes in the design variables. Second-order Taylor series expansion is usually not practicable due to the large computational effort involved in calculation of the second-order derivative matrices $H_j$. An exception to this is the common case of homogeneous displacement functions where Taylor series and the Binomial series become equivalent.

The advantage of using the Binomial series is that, unlike Taylor series, calculation of derivatives is not required. This makes the method more attractive in general applications where derivatives are not available. Calculation of each term of the Binomial series involves only forward and backward substitutions, if $K^*$ is given in the decomposed form of Eq. (2). Thus, the second-order terms can readily be calculated. In the case of first-order approximations, calculation of only $Br^*$ must be repeated for each trial design. This requires calculation of a single vector by forward and backward substitutions. Moreover, in the common case of homogeneous displacement functions, once the series coefficients are available, each redesign involves only calculation of simple products.[13] This makes the second-order approximations more attractive. However, it should be noted that the quality of the results obtained by second-order approximations of both Taylor series and the Binomial series might be insufficient in cases of large changes in the design variables.

As to the selection of the number of basis vectors to be considered in the proposed procedure, it has been noted that, in general, second-order approximations (three basis vectors) provide better results than first-order approximations (two basis vectors). However, in cases of poor selections of the first-order vectors, the second-order terms will not significantly improve the results. In cases of good selections of these vectors, on the other hand, the extra terms might not be necessary.

To evaluate the computational effort involved in the proposed procedure, compared with conventional methods, assume first-order approximations. Conventional series approximations require only calculation of the second basis vector [Eq. (26) or Eq. (27)]. The proposed procedure requires, in addition, introduction of the modified stiffness matrix $\boldsymbol{K}$, calculation of the products given by Eqs. (33) or Eqs. (34), determination of $\boldsymbol{y}$ by solving the set of $2 \times 2$ equations [Eq. (18)], and multiplying the two basis vectors by $\boldsymbol{y}$. Certainly, these operations increase the computational cost. However, the result is often considerably better approximations, particularly in cases of large changes in the design variables. It will be shown by the numerical examples that high-quality approximations can be obtained in cases where conventional series approximations provide meaningless results. Consequently, precise analyses, which involve more computational effort, are not required in cases where conventional approximations provide insufficient results.

Finally, conventional approximations may be viewed as a special case of the proposed procedure where $y = 1.0$ is selected. An additional advantage of the presented procedure is that, unlike conventional approximations, the errors involved in the approximations are evaluated by $\Delta \boldsymbol{R}$ and $Q$.

## Numerical Examples

In the following examples the displacements are evaluated for various changes in cross sections and geometrical variables. First- and second-order approximations will be considered by the following methods:

1) Taylor series expansion [Eq. (8)]
2) The proposed procedure, considering the criterion of Eq. (19) for the selection of $y$
3) The proposed procedure, considering the criterion of Eq. (24) for the selection of $y$

In all examples it is assumed that $W = I$ [Eq. (22)]. The selected basis vectors are as follows:

- Taylor series terms, for only cross-sectional variables (The Binomial series terms and Taylor series terms are identical in this case.)
- Binomial series terms, for geometrical variables (with or without cross-sectional variables)

### Five-Bar Truss

Consider the truss shown in Fig. 3a. The assumed modulus of elasticity is $E = 100$ and the four unknowns are horizontal ($r_h$) and vertical ($r_v$) displacements in joints A and B, respectively. Assume the initial cross-sectional areas $X^* = 1.0$ with the given displacements

$$r^{*T} = \{-55.78, 213.50, 44.22, 169.30\}$$

The generality of the proposed procedure and its effectiveness in solving problems with different types of variables is demonstrated by changing the initial geometry of Fig. 3a. Keeping the initial cross sections fixed, results obtained for the modified geometries shown in Figs. 3b and 3c by first-order approximations are summarized in Table 1. It can be observed that very good approximations have been obtained for these large changes in the geometry. Similar results have been obtained by the two different criteria of Eq. (19) and Eq. (24). This means that the proposed basis vectors ($r_1$ and $r_2$) are most effective for the two criteria of selecting $y$. It is instructive to note that the two modified geometries are similar in that geometry 3c is obtained from geometry 3b by multiplying all members' lengths by a factor of 0.5. Consequently, the displacements in geometry 3c are half the displacements in geometry 3b. This property of scaling of the members lengths, or scaling of the geometry, explains the high quality of the results obtained by the proposed procedure for geometrical changes.

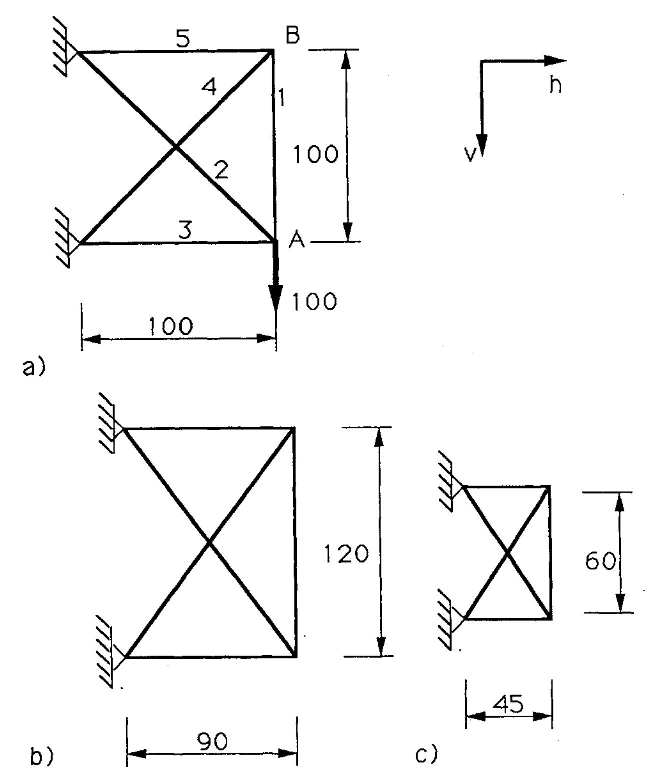


Fig. 3 Five-bar truss.

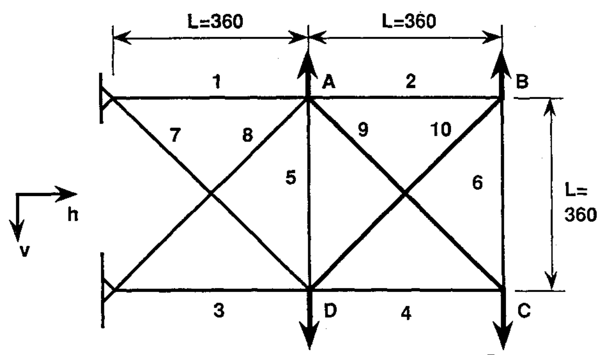


Fig. 4 Ten-bar truss.

Table 1 First-order approximations, five-bar truss, modified geometries

| Geometry | Method | Displacements | | | | $Q$ |
|---|---|---|---|---|---|---|
| 3b | Precise | −39.62 | 167.28 | 27.88 | 117.72 | — |
| | Eq. (24) | −39.43 | 167.33 | 27.40 | 117.87 | 0.65 |
| | Eq. (19) | −39.44 | 167.28 | 27.43 | 117.91 | 0.67 |
| 3c | Precise | −19.81 | 83.64 | 13.94 | 58.86 | — |
| | Eq. (24) | −19.72 | 83.66 | 13.70 | 58.93 | 0.65 |
| | Eq. (19) | −19.72 | 83.64 | 13.72 | 58.96 | 0.67 |

### Ten-Bar Truss

Consider the truss shown in Fig. 4 with 10 cross-sectional area design variables $X_i$ ($i = 1 \ldots 10$). The modulus of elasticity is 30,000 and the eight unknowns are horizontal ($r_h$) and vertical ($r_v$) displacements in joints A, B, C, and D, respectively. Two loading conditions have been considered:

Loading A: 100.0 (downward) at joints C, D

**Table 2 First-order approximations, 10-bar truss, changes of cross sections ($E = L = 1$)**

| Load | Method | Displacements | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| A | Precise | 19.519 | 53.174 | 23.491 | 115.55 | −26.509 | 120.52 | −20.481 | 57.537 |
| | Eq. (24) | 19.514 | 53.171 | 23.478 | 115.55 | −26.520 | 120.51 | −20.484 | 57.537 |
| | Eq. (19) | 19.515 | 53.172 | 23.479 | 115.55 | −26.521 | 120.52 | −20.484 | 57.538 |
| B | Precise | 19.038 | 50.991 | 21.981 | 113.07 | −28.019 | 123.00 | −20.962 | 59.719 |
| | Eq. (24) | 19.027 | 50.985 | 21.953 | 113.05 | −28.037 | 122.99 | −20.965 | 59.718 |
| | Eq. (19) | 19.029 | 50.987 | 21.956 | 113.06 | −28.040 | 122.99 | −20.968 | 59.719 |

**Table 3 First- and second-order approximations, 10-bar truss, optimal solutions ($E = L = 1$)**

| Load | Method | Terms | Displacements | | | | | | | | $Q$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| A | Precise | — | 25.0 | 75.0 | 40.5 | 184.4 | −50.0 | 200.0 | −25.0 | 75.0 | — |
| | Eq. (8) | 2 | −1180 | −2201 | −1223 | −4831 | 1218 | −4873 | 1219 | −2272 | — |
| | | 3 | 8468 | 13620 | 8615 | 31588 | −8649 | 31736 | −8739 | 13814 | — |
| | Eq. (24) | 2 | 21.1 | 68.8 | 31.6 | 160.7 | −40.1 | 171.1 | −22.7 | 77.0 | 790 |
| | | 3 | 22.8 | 67.4 | 35.6 | 172.5 | −47.6 | 185.4 | −24.6 | 75.9 | 103 |
| | Eq. (19) | 2 | 23.0 | 75.0 | 34.4 | 175.3 | −43.8 | 186.7 | −24.7 | 84.0 | 952 |
| | | 3 | 24.1 | 70.1 | 37.7 | 181.4 | −50.6 | 195.0 | −26.0 | 79.1 | 156 |
| B | Precise | — | 25.0 | 75.0 | 38.1 | 175.0 | −50.0 | 200.0 | −25.0 | 75.0 | — |
| | Eq. (8) | 2 | −1078 | −2189 | −1111 | −5031 | 1406 | −5125 | 1360 | −2469 | — |
| | | 3 | 7095 | 13729 | 7189 | 33615 | −11061 | 33887 | −10949 | 15209 | — |
| | Eq. (24) | 2 | 24.5 | 66.0 | 31.9 | 153.3 | −40.1 | 172.6 | −22.4 | 78.2 | 2119 |
| | | 3 | 21.8 | 63.5 | 31.0 | 156.8 | −47.2 | 180.5 | −24.5 | 75.5 | 166 |
| | Eq. (19) | 2 | 25.7 | 69.6 | 33.6 | 161.7 | −42.3 | 182.4 | −23.4 | 82.5 | 2345 |
| | | 3 | 22.6 | 66.1 | 32.3 | 164.0 | −50.0 | 189.0 | −26.0 | 78.6 | 283 |

Loading B: 150.0 (downward) at joints C, D and 50.0 (upward) at joints A, B

Assume the initial design $X^* = 1.0$ with the given displacements (for $E = L = 1.0$)

Loading A: $r^{*T} = \{195.4, 465.1, 235.5, 1054.2, -264.5, 1094.3, -204.6, 500.6\}$

Loading B: $r^{*T} = \{190.7, 447.3, 221.0, 1034.1, -279.0, 1114.4, -209.3, 518.3\}$

To illustrate the physical significance of the improved first-order approximations and the effectiveness of the proposed procedure, consider the modified design

$$X^T = \{10, 10, 10, 10, 8, 8, 8, 8, 8, 8\}$$

The displacements summarized in Table 2 show that excellent results have been obtained for these large changes (up to 900%) in the design variables. The high quality of the results can be explained by the scaling procedure. That is, the modified design is relatively close to the design line through $X^*$.

Consider the optimal designs[15]:

Loading A: $X^T = \{7.94, 0.1, 8.06, 3.94, 0.1, 0.1, 5.75, 5.57, 5.57, 0.1\}$

Loading B: $X^T = \{5.95, 0.1, 10.05, 3.95, 0.1, 2.05, 8.56, 2.75, 5.58, 0.1\}$

Results obtained for these very large changes in the cross sections (up to +905% and −90% simultaneously) by the various methods are shown in Table 3. It can be observed that solution by Taylor series [Eq. (8)] is meaningless. Specifically, the results obtained by second-order Taylor series approximations (three terms, or basis vectors, of the series) are worse than those obtained by first-order approximations (two terms of the series) since the series diverges. Relatively good results have been obtained by both methods of Eqs. (19) and (24). Considering three terms of the series (second-order approximations), the quality of the results is further improved. It has been noted that the results obtained by the criterion of Eq. (19) are often better than those obtained by Eq. (24), even in cases of larger $Q$ value.

#### Plane Frames

Consider the symmetric two-story frame shown in Fig. 5a with the initial design $EI = EA = 1.0$. The unknown displacements are horizontal translation (to the right), vertical

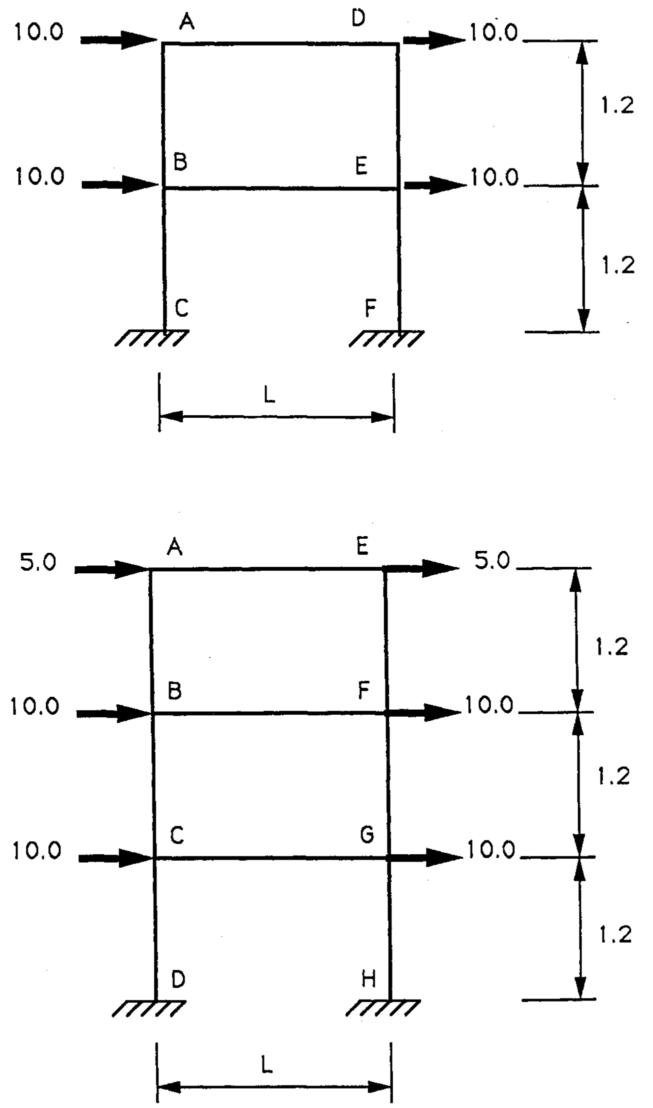


**Fig. 5 Frame examples.**

Table 4 First-order approximations, two-story frame, modified geometry and cross sections

| *EA* | Method | Displacements | | | | | |
|---|---|---|---|---|---|---|---|
| 1 | Precise | 13.00 | 5.76 | −6.79 | 5.18 | 3.90 | −5.04 |
| | Eq. (24) | 12.90 | 5.98 | −6.68 | 5.17 | 4.03 | −5.01 |
| | Eq. (19) | 12.95 | 5.98 | −6.69 | 5.19 | 4.01 | −5.01 |
| 10 | Precise | 12.14 | 1.72 | −4.37 | 5.48 | 1.25 | −5.53 |
| | Eq. (24) | 11.98 | 1.74 | −4.16 | 5.46 | 1.24 | −5.51 |
| | Eq. (19) | 12.09 | 1.73 | −4.21 | 5.50 | 1.24 | −5.58 |
| ∞ | Precise | 10.10 | 0 | −2.91 | 4.90 | 0 | −4.59 |
| | Eq. (24) | 9.93 | 0 | −2.67 | 4.88 | 0 | −4.54 |
| | Eq. (19) | 10.04 | 0 | −2.73 | 4.91 | 0 | −4.62 |

Table 5 First-order approximations, three-story frame, modified geometries

| *L* | Method | Displacements | | | | | |
|---|---|---|---|---|---|---|---|
| 2.0 | Precise | 19.16 | −1.63 | 15.12 | −3.91 | 7.11 | −5.84 |
| 4.0 | Precise | 29.02 | −3.90 | 21.57 | −7.32 | 9.31 | −9.51 |
| | Eq. (24) | 28.56 | −3.37 | 21.34 | −6.98 | 9.29 | −9.52 |
| | Eq. (19) | 28.67 | −3.46 | 21.61 | −7.12 | 9.38 | −9.70 |
| 1.0 | Precise | 13.31 | −0.68 | 10.98 | −2.00 | 5.61 | −3.35 |
| | Eq. (24) | 13.09 | −0.50 | 10.94 | −1.90 | 5.61 | −3.41 |
| | Eq. (19) | 13.18 | −0.52 | 11.00 | −1.93 | 5.63 | −3.46 |

translation (upward), and rotation (counterclockwise) in joints A and B, respectively. Assuming the initial geometry $L = 2.0$ (where $L$ is the span), the initial displacements are

$$r^{*T} = (21.36, 10.50, -11.14, 8.06, 8.19, -9.83\}$$

Assuming the modified geometry $L = 4.0$, results obtained for three cases of $EA$ by the proposed first-order approximations are summarized in Table 4.

Consider the symmetric three-story frame shown in Fig. 5b. Assume $EI = 1.0$ for all members and consider only flexural deformations ($EA = \infty$). The unknown displacements are horizontal translation (to the right) and rotation (counterclockwise) in joints A, B, and C, respectively. Assuming the initial geometry $L = 2.0$, results obtained for the modified geometries $L = 4.0$ and $L = 1.0$ by first-order approximations are summarized in Table 5. It can be observed that very good results have been obtained by both criteria of Eqs. (19) and (24).

### Thirteen-Bar Truss

Consider the 13-bar truss with the initial geometry and loading shown in Fig. 6a. The modulus of elasticity is 10,000, the initial cross sections are $X^* = 1.0$, and the unknown displacements are $r_h$ and $r_v$ in joints B, C, D, F, G, and H, respectively.

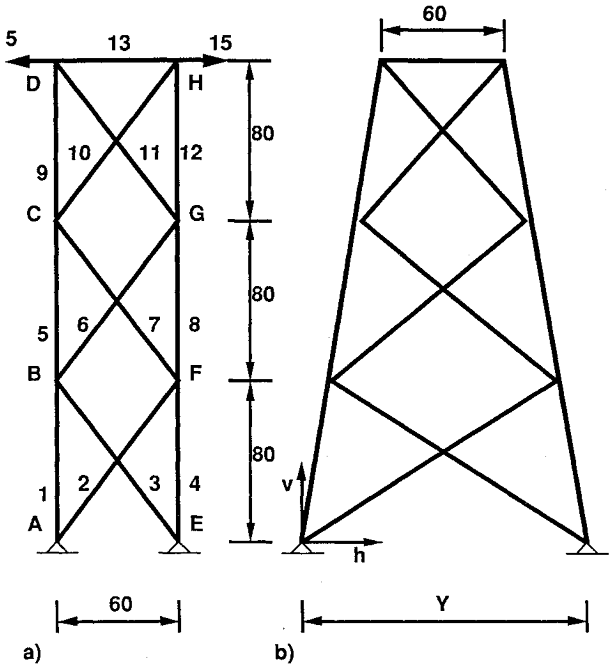


Fig. 6 Thirteen-bar truss.

Assume first the following modified cross-sectional areas

$$X^T = \{10, 10, 10, 10, 8, 8, 8, 8, 6, 6, 6, 6, 6\}$$

Results obtained for these large changes in cross sections (up to 900%) by various approximate methods are shown in Table 6. It can be observed that:

1) Results obtained by conventional first- and second-order series expansion [Eq. (8) or Eq. (13), which are equivalent in this case] are meaningless. Once again, the series diverges due to the large changes in the design.

2) Very good results have been obtained by first- and second-order approximations for both methods of Eqs. (19) and (24).

Consider a single geometric variable $Y$ representing the span (Fig. 6b). Assume the optimal design for this example[12] $Y = 153$, and the modified cross-sectional areas $X$ as in the previous case.

Results obtained for these large changes in both geometry (155% in $Y$) and cross sections (up to 900%) by the proposed solution procedure are shown in Table 7. Improved results have been obtained for second-order approximations (three terms) by both methods of Eqs. (19) and (24). Specifically, assuming three terms for the method of Eq. (19), the errors in most displacements are less than 5%.

Table 6 First- and second-order approximations, 13-bar truss, modified cross sections

| Displacement number | Eq. (8) or (13) | | Eq. (24) | | Eq. (19) | | Precise method |
|---|---|---|---|---|---|---|---|
| | 2 Terms | 3 Terms | 2 Terms | 3 Terms | 2 Terms | 3 Terms | |
| 1 | −3.90 | 35.7 | 0.051 | 0.048 | 0.048 | 0.048 | 0.048 |
| 2 | −2.12 | 19.3 | 0.028 | 0.026 | 0.026 | 0.026 | 0.026 |
| 3 | −11.85 | 103.8 | 0.178 | 0.167 | 0.168 | 0.167 | 0.167 |
| 4 | −3.1 | 26.3 | 0.050 | 0.047 | 0.047 | 0.047 | 0.047 |
| 5 | −20.7 | 177.2 | 0.338 | 0.319 | 0.320 | 0.319 | 0.319 |
| 6 | −3.3 | 27.4 | 0.058 | 0.055 | 0.055 | 0.055 | 0.055 |
| 7 | −4.0 | 36.5 | 0.054 | 0.051 | 0.050 | 0.051 | 0.051 |
| 8 | 2.1 | −19.6 | −0.029 | −0.027 | −0.027 | −0.027 | −0.027 |
| 9 | −11.7 | 102.5 | 0.173 | 0.162 | 0.162 | 0.162 | 0.162 |
| 10 | 3.1 | −26.4 | −0.050 | −0.046 | −0.047 | −0.047 | −0.047 |
| 11 | −21.0 | 178.4 | 0.348 | 0.329 | 0.329 | 0.329 | 0.329 |
| 12 | 3.3 | −27.5 | −0.059 | −0.056 | −0.056 | −0.056 | −0.056 |

Table 7 First- and second-order approximations, 13-bar truss, modified geometry and cross sections

| Displacement number | Precise method | Eq. (24) | | | Eq. (19) | | |
|---|---|---|---|---|---|---|---|
| | | 2 Terms | 3 Terms | Error, % | 2 Terms | 3 Terms | Error, % |
| 1 | 0.0098 | 0.0043 | 0.0105 | 7 | 0.0089 | 0.0112 | 14 |
| 2 | 0.0101 | 0.0059 | 0.0091 | 10 | 0.0084 | 0.0099 | 2 |
| 3 | 0.0400 | 0.0230 | 0.0364 | 9 | 0.0378 | 0.0396 | 1 |
| 4 | 0.0162 | 0.0105 | 0.0142 | 12 | 0.0146 | 0.0155 | 4 |
| 5 | 0.0811 | 0.0518 | 0.0726 | 10 | 0.0788 | 0.0795 | 2 |
| 6 | 0.0134 | 0.0095 | 0.0114 | 15 | 0.0140 | 0.0127 | 5 |
| 7 | 0.0114 | 0.0061 | 0.0108 | 5 | 0.0108 | 0.0117 | 3 |
| 8 | −0.0101 | −0.0063 | −0.0089 | 12 | −0.0088 | −0.0096 | 5 |
| 9 | 0.0360 | 0.0185 | 0.0339 | 6 | 0.0330 | 0.0367 | 2 |
| 10 | −0.0166 | −0.0104 | −0.0141 | 15 | −0.0145 | −0.0154 | 7 |
| 11 | 0.0902 | 0.0603 | 0.0817 | 9 | 0.0879 | 0.0895 | 1 |
| 12 | −0.0133 | −0.0102 | −0.0120 | 10 | −0.0147 | 0.0133 | 0 |

## Conclusions

Approximations of the structural behavior in terms of the design variables are essential in optimization of large-scale structures, where the time-consuming analysis must be repeated many times. A major problem is that the quality of the commonly used approximations might be insufficient, particularly in cases of large changes in the design.

A general solution procedure for effective approximations is proposed in this study. The quality of displacement approximations can greatly be improved by combining the computed terms of a series expansion, used as high-quality basis vectors, with the coefficients of a reduced basis expression. It is shown that the latter coefficients can readily be determined such that the errors in satisfying the analysis equations are minimized. An alternative criterion for selecting these coefficients, based on satisfying a reduced set of equations, is demonstrated. The two criteria are compared in terms of the quality of the results.

It is shown that first-order approximations for the proposed procedure can be introduced by combining scaling of the initial stiffness matrix and scaling of a fictitious set of loads. Integrating these two types of scaling, the approximate displacements can be expressed in a reduced basis form as functions of two coefficients. The accuracy of the presented first-order approximations is often sufficient and calculation of higher order terms is not necessary.

The proposed approach is general and can be applied with different types of design variables (i.e., geometrical variables, cross-sectional varibles, etc.). It can be used with various finite element programs considering different versions, such as:

- Various selections of the basis vectors [i.e., Eq. (8) or Eq. (13)]
- Various criteria for selecting the parameters $y$ [i.e., Eq. (19) or Eq. (24)]

The computational effort involved in the proposed procedure is larger, compared with conventional Taylor series or Binomial series approximations. However, the result is considerably better approximations, particularly in cases of large changes in the design variables. Consequently, precise analyses that involve more computational effort are not required in cases where conventional approximations provide insufficient results. It has been noted that conventional approximations may be viewed as a special case of the proposed procedure where $y = 1.0$ is selected. An additional advantage of the presented procedure is that, unlike conventional approximations, the errors involved in the approximations can be evaluated by $\Delta \boldsymbol{R}$ and $Q$.

Several simple examples illustrate the solution methodology and the effectiveness of the proposed procedure. Specifically, the following observations have been made:

- Good results have been obtained for very large changes in the design with a relatively small computational effort.
- Similar results have been obtained by the two different criteria of Eqs. (19) and (24). This means that the proposed basis vectors are most effective for various criteria of selecting $y$.

In summary, the proposed solution procedure is a powerful tool to achieve efficient and high-quality approximations. It also provides insight and better understanding of the behavior of structural models.

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